

Some remarks on sets with small quotient set *

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Annotation.

We prove, in particular, that for any finite set $A \subset \mathbb{R}$ with $|A/A| \ll |A|$ one has $|A-A| \gg |A|^{5/3-o(1)}$. Also we show that $|3A| \gg |A|^{2-o(1)}$ in the case.

1 Introduction

Let $A, B \subset \mathbb{R}$ be finite sets. Define the *sum set*, the *product set* and the *quotient set* of A and B as

$$A+B := \{a+b : a \in A, b \in B\},$$

$$AB := \{ab : a \in A, b \in B\},$$

and

$$A/B := \{a/b : a \in A, b \in B, b \neq 0\},$$

correspondingly. Sometimes we write kA for multiple sumsets, difference and so on, e.g. $A+A+A=3A$. The Erdős-Szemerédi conjecture [5] says that for any $\epsilon > 0$ one has

$$\max\{|A+A|, |AA|\} \gg |A|^{2-\epsilon}. \quad (1)$$

Modern bounds concerning the conjecture can be found in [20], [9], [10]. The first interesting case of Conjecture (1) was proved in [4], see also [20], namely

$$|A+A| \ll |A| \quad \text{or} \quad |A-A| \ll |A| \implies |AA| \gg |A|^{2-\epsilon} \quad \text{or} \quad |A/A| \gg |A|^{2-\epsilon}.$$

The opposite situation is wide open and it is called sometimes a *weak Erdős-Szemerédi Conjecture* [13]. So, it is unknown

$$|AA| \ll |A| \quad \text{or} \quad |A/A| \ll |A| \implies |A+A| \gg |A|^{2-\epsilon} \quad \text{or} \quad |A-A| \gg |A|^{2-\epsilon} ? \quad (2)$$

The best current lower bounds on the size of sumsets of sets A with small AA or A/A are contained in [9], [10]. As for difference sets it was proved in [19], [7] that

$$|AA| \ll |A| \implies |A-A| \gg |A|^{14/11-\epsilon} \quad \text{and} \quad |A/A| \ll |A| \implies |A-A| \gg |A|^{8/5-\epsilon}.$$

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The integer situation was considered in [2] (in the paper M.-C. Chang has deal with the case of multiple sumsets as well).

Let us formulate the first main result of our paper (see Theorem 10 below).

Theorem 1 *Let $A \subset \mathbb{R}$ be a finite set. Then*

$$|A/A| \ll |A| \implies |A - A| \gg |A|^{5/3-\epsilon}.$$

Our method uses some ideas from the higher energies, see [14] and has some intersections with [19]. The main new ingredient is the following observation. Let us suppose that there is a family of finite (multidimensional) sets A_j , $j = 1, \dots, n$ and we want to obtain a lower bound for $\bigcup_{j=1}^n A_j$ better than $\max_j |A_j|$. Let us assume the contrary and the first simple model situation is $A_1 = \dots = A_n$, so we need to separate from the case at least. Suppose that for any j there is a map (projection) π_j associated with each set A_j . We should think about the maps π_j as about "different" maps somehow (in particular they cannot coincide). More precisely, if one is able to prove that $\bigcup_{j=1}^n \pi_i(A_j)$ is strictly bigger than $\max_j |\pi_i(A_j)|$ then it cannot be the case $A_1 = \dots = A_n$ and hence $\bigcup_{j=1}^n A_j$ should be large. For more precise formulation see the proof of Theorem 10.

Our second main result shows that Conjecture (2) holds if one considers $A + A + A$ or $A + A - A$, see Theorem 12 below.

Theorem 2 *Let $A \subset \mathbb{R}$ be a finite set, and $|AA| \ll |A|$ or $|A/A| \ll |A|$. Then for any $\alpha, \beta \neq 0$ one has*

$$|A + \alpha A + \beta A| \gg \frac{|A|^2}{\log |A|}.$$

Theorem 2 is an analog of main Theorem 1 from [17] and it is proved by a similar method. Also we study different properties of sets with small product/quotient set, see section 5.

The best results for multiple sumsets kA , $k \rightarrow \infty$ of sets A with small product/quotient set can be found in [1], see also our remarks in section 5.

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2 Notation

Let \mathbf{G} be an abelian group. In this paper we use the same letter to denote a set $S \subseteq \mathbf{G}$ and its characteristic function $S : \mathbf{G} \rightarrow \{0, 1\}$. By $|S|$ denote the cardinality of S .

Let $f, g : \mathbf{G} \rightarrow \mathbb{C}$ be two functions. Put

$$(f * g)(x) := \sum_{y \in \mathbf{G}} f(y)g(x - y) \quad \text{and} \quad (f \circ g)(x) := \sum_{y \in \mathbf{G}} f(y)g(y + x). \quad (3)$$

By $E^+(A, B)$ denote the *additive energy* of two sets $A, B \subseteq \mathbf{G}$ (see e.g. [22]), that is

$$E^+(A, B) = |\{a_1 + b_1 = a_2 + b_2 : a_1, a_2 \in A, b_1, b_2 \in B\}|.$$

If $A = B$ we simply write $E^+(A)$ instead of $E^+(A, A)$. Clearly,

$$E^+(A, B) = \sum_x (A * B)(x)^2 = \sum_x (A \circ B)(x)^2 = \sum_x (A \circ A)(x)(B \circ B)(x).$$

Note also that

$$E^+(A, B) \leq \min\{|A|^2|B|, |B|^2|A|, |A|^{3/2}|B|^{3/2}\}. \quad (4)$$

More generally (see [14]), for $k \geq 2$ put

$$E_k^+(A) = |\{a_1 - a'_1 = a_2 - a'_2 = \dots = a_k - a'_k : a_i, a'_i \in A\}|.$$

Thus, $E^+(A) = E_2^+(A)$.

In the same way define the *multiplicative energy* of two sets $A, B \subseteq \mathbf{G}$

$$E^\times(A, B) = |\{a_1 b_1 = a_2 b_2 : a_1, a_2 \in A, b_1, b_2 \in B\}|$$

and, similarly, $E_k^\times(A)$. Certainly, the multiplicative energy $E^\times(A, B)$ can be expressed in terms of multiplicative convolutions, as in (3). We often use the notation

$$A_\lambda = A_\lambda^\times = A \cap (\lambda^{-1}A)$$

for any $\lambda \in A/A$. Hence

$$E^\times(A) = \sum_{\lambda \in A/A} |A_\lambda|^2.$$

For given integer $k \geq 2$, a fixed vector $\vec{\lambda} = (\lambda_1, \dots, \lambda_{k-1})$ and a set A put

$$\Delta_{\vec{\lambda}}(A) = \{(\lambda_1 a, \lambda_2 a, \dots, \lambda_{k-1} a, a) : a \in A\} \subseteq A^k.$$

All logarithms are base 2. Signs \ll and \gg are the usual Vinogradov's symbols. Having a set A , we write $a \lesssim b$ or $b \gtrsim a$ if $a = O(b \cdot \log^c |A|)$, $c > 0$. For any given prime p denote by \mathbb{F}_p the finite prime field and put $\mathbb{F}_p^* = \mathbb{F}_p \setminus \{0\}$.

3 Preliminaries

Again, let $\mathbf{G} = (\mathbf{G}, +)$ be an abelian group with the group operation $+$. We begin with the famous Plünnecke–Ruzsa inequality (see [22], e.g.).

Lemma 3 *Let $A, B \subseteq \mathbf{G}$ be two finite sets, $|A + B| \leq K|A|$. Then for all positive integers n, m the following holds*

$$|nB - mB| \leq K^{n+m}|A|. \quad (5)$$

Further, for any $0 < \delta < 1$ there is $X \subseteq A$ such that $|X| \geq (1 - \delta)|A|$ and for any integer k one has

$$|X + kB| \leq (K/\delta)^k |X|. \quad (6)$$

We need a simple lemma.

Lemma 4 *Let $A \subset \mathbb{R}$ be a finite set. Then there is z such that*

$$\sum_{x \in zA} |zA \cap x(zA)| \gg \frac{E^\times(A)}{|A|}.$$

Proof. Without loss of generality one can suppose that $0 \notin A$. We have

$$E^\times(A) = \sum_x |A \cap xA|^2 \leq 2 \sum_{x : |A \cap xA| > E^\times(A)/(2|A|^2)} |A \cap xA|^2.$$

Thus, putting $\Delta = E^\times(A)/(2|A|^2)$ and P equals

$$P = \{x : \Delta < |A \cap xA|\},$$

we get $|P|\Delta^2 \gg E^\times(A)$. Let $A' = \{x \in A : |P \cap x^{-1}A| \geq 2^{-1}\Delta|P|\}$. Because of $P = P^{-1}$, we have

$$\Delta|P| < \sum_{x \in P} |A \cap xA| = \sum_{x \in A} |P \cap xA^{-1}| = \sum_{x \in A} |P \cap x^{-1}A| \leq 2 \sum_{x \in A'} |P \cap x^{-1}A|.$$

In other words,

$$\Delta|P| \ll \sum_{x \in A'} |P \cap x^{-1}A| = \sum_{x \in A} |P \cap x^{-1}A'|.$$

It follows that there is $x \in A$ with $|P \cap x^{-1}A'| \gg \Delta|P|/|A|$. Put $W = x(P \cap x^{-1}A') \subseteq A' \subseteq A$ and note that

$$E^\times(A)|A|^{-1} \ll \Delta^2|P||A|^{-1} \ll |W|\Delta < \sum_{y \in x^{-1}W} |A \cap yA| \leq \sum_{y \in x^{-1}A} |x^{-1}A \cap y(x^{-1}A)|$$

as required. \square

The method of the paper relies on the famous Szemerédi–Trotter Theorem [21], see also [22]. Let us recall the definitions.

We call a set \mathcal{L} of continuous plane curves a *pseudo-line system* if any two members of \mathcal{L} are determined by two points. Define the *number of indices* $\mathcal{I}(\mathcal{P}, \mathcal{L})$ between points and pseudo-lines as $\mathcal{I}(\mathcal{P}, \mathcal{L}) = |\{(p, l) \in \mathcal{P} \times \mathcal{L} : p \in l\}|$.

Theorem 5 *Let \mathcal{P} be a set of points and let \mathcal{L} be a pseudo-line system. Then*

$$\mathcal{I}(\mathcal{P}, \mathcal{L}) \ll |\mathcal{P}|^{2/3}|\mathcal{L}|^{2/3} + |\mathcal{P}| + |\mathcal{L}|.$$

A simple consequence of Theorem 5 was obtained in [16], see Lemma 7.

Lemma 6 *Let $A \subset \mathbb{R}$ be a finite set. Put $M(A)$ equals*

$$M(A) := \min_{B \neq \emptyset} \frac{|AB|^2}{|A||B|}. \quad (7)$$

Then

$$E_3^+(A) \ll M(A)|A|^3 \log |A|. \quad (8)$$

Also we need a result from [11]. Let $T(A)$ be the number of *collinear triples* in $A \times A$.

Theorem 7 *Let $A \subset \mathbb{R}$ be a finite set. Then*

$$T(A) \ll |A|^4 \log |A|.$$

More generally, for three finite sets $A, B, C \subset \mathbb{R}$ put $T(A, B, C)$ be the number of collinear triples in $A \times A, B \times B, C \times C$, correspondingly. Clearly, the quantity $T(A, B, C)$ is symmetric on all its variables. Further, it is easy to see that

$$T(A, B, C) = \left| \left\{ \frac{c_1 - a_1}{b_1 - a_1} = \frac{c_2 - a_2}{b_2 - a_2} : a_1, a_2 \in A, b_1, b_2 \in B, c_1, c_2 \in C \right\} \right| + 2|A \cap B \cap C||A||B||C|,$$

and

$$T(A, B, B) = \sum_{a_1, a_2 \in A} E^\times(B - a_1, B - a_2). \quad (9)$$

Corollary 8 *Let $A, B \subset \mathbb{R}$ be two finite sets, $|B| \leq |A|$. Then*

$$T(A, B, B) \ll |A|^2 |B|^2 \log |B|,$$

and for any finite $A_1, A_2 \subset \mathbb{R}$, $|B| \leq |A_1|, |A_2|$ one has

$$T(A_1, A_2, B) \ll |A_1|^2 |A_2|^2 \log |B|.$$

Proof. Split A onto $t \ll |A|/|B|$ parts B_j of size at most $|B|$. Then, using Theorem 7, we get

$$T(A, B, B) \leq \sum_{i,j=1}^t T(B_i \times B_j, B, B) \ll t^2 |B|^4 \log |B| \ll |A|^2 |B|^2 \log |B|$$

as required. The second bound follows similarly. This completes the proof. \square

We need a result from [12], which is a consequence of the main theorem from [13].

Theorem 9 *Let $A, B, C \subseteq \mathbb{F}_p$, and let $M = \max(|A|, |BC|)$. Suppose that $|A||B||BC| \ll p^2$. Then*

$$E^+(A, C) \ll (|A||BC|)^{3/2} |B|^{-1/2} + M|A||BC||B|^{-1}. \quad (10)$$

4 The proof of the main result

Now let us obtain a lower bound for the difference set of sets with small quotient set.

Theorem 10 *Let $A \subset \mathbb{R}$ be a finite set. Then*

$$|A - A|^6 |A/A|^{13} \gtrsim |A|^{23}. \quad (11)$$

In particular, if $|A/A| \ll |A|$ then $|A - A| \gtrsim |A|^{5/3}$.

Proof. Let $\Pi = A/A$. Put M equals $|\Pi|/|A|$. Without loss of generality one can suppose that $0 \notin A$. Let $D = A - A$. Let also $\mathcal{P} = D \times D$. Then for any $\lambda \in \Pi$ one has

$$Q_\lambda := A \times A_\lambda - \Delta_\lambda(A_\lambda) \subseteq \mathcal{P}.$$

Further, for an arbitrary $\lambda \in \Pi$ consider a projection $\pi_\lambda(x, y) = x - \lambda y$. Then, it is easy to check that $\pi_\lambda(Q_\lambda) \subseteq D$. In other words, if we denote by \mathcal{L}_λ the set of all lines of the form $\{(x, y) : x - \lambda y = c\}$, intersecting the set Q_λ , we obtain that $|\mathcal{L}_\lambda| \leq |D|$. Finally, take any set $\Lambda \subseteq \Pi$, $\Lambda = \Lambda^{-1}$, and put $\mathcal{L} = \bigsqcup_{\lambda \in \Lambda} \mathcal{L}_\lambda$. It follows that

$$|\mathcal{L}| = \sum_{\lambda \in \Lambda} |\mathcal{L}_\lambda| \leq |D| |\Lambda|. \quad (12)$$

By the construction the number of indices $\mathcal{I}(\mathcal{P}, \mathcal{L})$ between points \mathcal{P} and lines \mathcal{L} is at least $\mathcal{I}(\mathcal{P}, \mathcal{L}) \geq \sum_{\lambda \in \Lambda} |Q_\lambda|$. Applying Szemerédi–Trotter Theorem 5, using formula (12), and making simple calculations, we get

$$\sum_{\lambda \in \Lambda} |Q_\lambda| \leq \mathcal{I}(\mathcal{P}, \mathcal{L}) \ll (|\mathcal{L}| |\mathcal{P}|)^{2/3} + |\mathcal{L}| + |\mathcal{P}| \ll |D|^2 |\Lambda|^{2/3}. \quad (13)$$

Hence, our task is to find a good lower bound for the sum $\sum_{\lambda \in \Lambda} |Q_\lambda|$. For any $\lambda \in \Pi$, we have

$$|A| |A_\lambda|^2 = \sum_{x, y} \sum_z A_\lambda(z) A(\lambda z + x) A_\lambda(z + y) = \sum_{(x, y) \in Q_\lambda} \sum_z A_\lambda(z) A(\lambda z + x) A_\lambda(z + y),$$

and, thus, by the Cauchy–Schwarz inequality, we get

$$|A| |A_\lambda|^2 \leq |Q_\lambda|^{1/2} \cdot \left(\sum_{x, y} \left(\sum_z A_\lambda(z) A(\lambda z + x) A_\lambda(z + y) \right)^2 \right)^{1/2}.$$

Summing over $\lambda \in \Lambda$ and applying the Cauchy–Schwarz inequality once more time, we obtain

$$\begin{aligned} |A|^2 (\mathbb{E}_\Lambda^\times(A))^2 &:= |A|^2 \left(\sum_{\lambda \in \Lambda} |A_\lambda|^2 \right)^2 \leq \sum_{\lambda \in \Lambda} |Q_\lambda| \cdot \sum_{\lambda \in \Lambda} \sum_{x, y} \left(\sum_z A_\lambda(z) A(\lambda z + x) A_\lambda(z + y) \right)^2 = \\ &= \sum_{\lambda \in \Lambda} |Q_\lambda| \cdot \sum_{\lambda \in \Lambda} \sum_w (A_\lambda \circ A_\lambda)^2(w) (A \circ A)(\lambda w) = \sigma_1 \cdot \sigma_2. \end{aligned} \quad (14)$$

Let us estimate the sum σ_2 . Putting $\tilde{A}_\lambda = A \cap \lambda A$, we see that by the Hölder inequality the following holds

$$\begin{aligned} \sigma_2 &= \sum_{\lambda \in \Lambda} \sum_w (A_\lambda \circ A_\lambda)^2(w/\lambda)(A \circ A)(w) = \sum_{\lambda \in \Lambda} \sum_w (\tilde{A}_\lambda \circ \tilde{A}_\lambda)^2(w)(A \circ A)(w) \leq \\ &\leq (\mathbf{E}_3^+(A))^{1/3} \cdot \sum_{\lambda \in \Lambda} (\mathbf{E}_3^+(\tilde{A}_\lambda))^{2/3}. \end{aligned}$$

Put $\Lambda \subseteq \Pi$, $\Lambda = \Lambda^{-1}$ such that

$$\frac{|A|^3}{M} \leq \mathbf{E}^\times(A) \lesssim \mathbf{E}_\Lambda(A). \quad (15)$$

The first bound in (15) is just the Cauchy–Schwarz inequality (4) and the existence of the set Λ follows from the simple pigeonholing. In particular, it follows that $|\tilde{A}_\lambda| = |A_\lambda| \gg |A|/M$ and hence $|\Lambda| \ll M|A|$. Because of $|A/A| \leq M|A|$, we clearly have $M(A) \leq M^2$. Applying Lemma 6 and the notation from (7) for the set A as well for the sets \tilde{A}_λ , we get

$$\sigma_2 \lesssim M^{2/3}|A| \cdot \sum_{\lambda \in \Lambda} M^{2/3}(\tilde{A}_\lambda)|\tilde{A}_\lambda|^2.$$

It is easy to see that

$$M(\tilde{A}_\lambda) \leq \frac{|A\tilde{A}_\lambda|^2}{|A||\tilde{A}_\lambda|} \leq \frac{|AA|^2}{|A||\tilde{A}_\lambda|} \leq \frac{M^2|A|}{|\tilde{A}_\lambda|} \leq M^3, \quad (16)$$

and hence

$$\sigma_2 \lesssim M^{8/3}|A| \cdot \mathbf{E}_\Lambda^\times(A).$$

Here we have used the fact $\Lambda = \Lambda^{-1}$. Returning to (15) and using the Cauchy–Schwarz inequality, we get

$$\sum_{\lambda \in \Lambda} |Q_\lambda| \gtrsim \frac{|A|^4}{M^{11/3}}.$$

Combining the last bound with (13), we obtain

$$\frac{|A|^{12}}{M^{11}} \lesssim |D|^6 |\Lambda|^2 \leq M^2 |A|^2 |D|^6$$

as required. \square

Remark 11 Careful analysis of the proof (e.g. one should use the estimate $M(\tilde{A}_\lambda) \leq M^2|A|/|\tilde{A}_\lambda|$ from (16)) shows that we have obtained an upper bound for the higher energy $\mathbf{E}_8^\times(A)$. Namely,

$$|A|^7 \mathbf{E}_8^\times(A) \lesssim |A/A|^6 |A - A|^6.$$

The last bound is always better than Elekes' inequality for quotient sets [3]

$$|A|^5 \ll |A/A|^2 |A \pm A|^2.$$

Now let us prove our second main result, which corresponds to the main theorem from [17].

Theorem 12 *Let $A \subset \mathbb{R}$ be a finite set, and $|AA| \leq M|A|$ or $|A/A| \leq M|A|$. Then for any $\alpha \neq 0$ one has*

$$E^\times(A + \alpha) \ll M^4 |A|^2 \log |A|. \quad (17)$$

In particular,

$$|AA + A + A| \geq |(A + 1)(A + 1)| \gg \frac{|A|^2}{M^4 \log |A|}. \quad (18)$$

Finally, for any $\alpha, \beta \neq 0$ the following holds

$$|A + \alpha A + \beta A| \gg \frac{|A|^2}{M^6 \log |A|}. \quad (19)$$

Proof. Without loss of generality one can suppose that $0 \notin A$. Let $\Pi = AA$, $Q = A/A$. Applying the second estimate of Corollary 8 with $B = -\alpha A$, $A_1 = A_2 = \Pi$ as well as formula (9), we get

$$\sum_{a, a' \in A} E^\times(\Pi + \alpha a, \Pi + \alpha a') \ll M^4 |A|^4 \log |A|.$$

Thus there are $a, a' \in A$ such that $E^\times(\Pi + \alpha a, \Pi + \alpha a') \ll M^4 |A|^2 \log |A|$. In other words,

$$E^\times(\Pi/a + \alpha, \Pi/a' + \alpha) = E^\times(\Pi + \alpha a, \Pi + \alpha a') \ll M^4 |A|^2 \log |A|.$$

Clearly, $A \subseteq \Pi/a$, $A \subseteq \Pi/a'$ and hence $E^\times(A + \alpha) \ll M^4 |A|^2 \log |A|$. To obtain the same estimate with Q just note that for any $a \in A$ one has $A \subseteq Qa$ and apply the same arguments with $B = -\alpha A^{-1}$. Further, by estimate (17) with $\alpha = 1$ and bound (4), we have

$$|AA + A + A| = |AA + A + A + 1| \geq |(A + 1)(A + 1)| \gg \frac{|A|^2}{M^4 \log |A|}$$

and (18) follows.

It remains to prove (19). Using Lemma 4, we find z such that

$$\frac{|A|^2}{M} \leq \sum_{\lambda \in zA} |(zA) \cap \lambda^{-1}(zA)|.$$

With some abuse of the notation redefine A to be zA and thus, we have

$$\frac{|A|^2}{M} \leq \sum_{\lambda \in A} |A \cap \lambda^{-1}A| = \sum_{\lambda \in A} |A_\lambda|. \quad (20)$$

Further, using the previous arguments, we get

$$\sum_{a, a' \in A} E^\times(Q + \alpha/a, Q + \beta/a') \ll M^4 |A|^4 \log |A|, \quad (21)$$

and

$$\sum_{a, a' \in A} \mathbf{E}^\times(\Pi + \alpha a, \Pi + \beta a') \ll M^4 |A|^4 \log |A|. \quad (22)$$

Let us consider the case of the set Q , the second situation is similar. From (21), we see that there are $a, a' \in A$ such that

$$\begin{aligned} \sigma &:= |\{(q_1 a + \alpha)(q'_1 a' + \beta) = (q_2 a + \alpha)(q'_2 a' + \beta) : q_1, q'_1, q_2, q'_2 \in Q\}| = \\ &= \mathbf{E}^\times(Q + \alpha/a, Q + \beta/a') \ll M^4 |A|^2 \log |A|. \end{aligned}$$

Using the inclusion $A \subseteq Qa$, $a \in A$ once more time, it is easy to check that

$$\sigma \geq |\{(a_1 + \alpha)(a'_1 + \beta) = (a_2 + \alpha)(a'_2 + \beta) : a_1, a_2 \in A, a'_1 \in A_{a_1}, a'_2 \in A_{a_2}\}| = \sum_x n^2(x),$$

where

$$n(x) = |\{(a_1 + \alpha)(a'_1 + \beta) = x : a_1 \in A, a'_1 \in A_{a_1}\}|.$$

Clearly, the support of the function $n(x)$ is $A + \alpha A + \beta A + \alpha\beta$. Using the Cauchy–Schwarz inequality and bound (20), we obtain

$$\begin{aligned} \frac{|A|^4}{M^2} &\leq \left(\sum_{\lambda \in A} |A_\lambda| \right)^2 = \left(\sum_x n(x) \right)^2 \leq |A + \alpha A + \beta A| \cdot \sum_x n^2(x) \leq \\ &\leq |A + \alpha A + \beta A| \cdot \sigma \ll |A + \alpha A + \beta A| \cdot M^4 |A|^2 \log |A| \end{aligned}$$

as required. \square

5 Further remarks

Now let us make some further remarks on sets with small quotient/product set. First of all let us say a few words about multiple sumsets kA of sets A with small multiplicative doubling. As was noted before when k tends to infinity the best results in the direction were obtained in [1]. For small $k > 3$ another methods work. We follow the arguments from [8] with some modifications.

Suppose that $A \subset \mathbf{G}$ is a finite set, where \mathbf{G} is an abelian group with the group operation \times . Put $\|A\|_{\mathcal{U}^k}$ to be Gowers non-normalized k th-norm [6] of the characteristic function of A (in multiplicative form), see, say [15]. For example, $\|A\|_{\mathcal{U}^2} = \mathbf{E}^\times(A)$ is the multiplicative energy of A and

$$\|A\|_{\mathcal{U}^3} = \sum_{\lambda \in A/A} \mathbf{E}^\times(A_\lambda).$$

Moreover, the induction property for Gowers norms holds, see [6]

$$\|A\|_{\mathcal{U}^{k+1}} = \sum_{\lambda \in A/A} \|A_\lambda\|_{\mathcal{U}^k}. \quad (23)$$

It was proved in [6] that k th-norms of the characteristic function of any set are connected to each other. In [15] the author shows that the connection for the non-normalized norms does not depend on the size of \mathbf{G} . Here we formulate a particular case of Proposition 35 from [15], which connects $\|A\|_{\mathcal{U}^k}$ and $\|A\|_{\mathcal{U}^2}$, see Remark 36 here.

Lemma 13 *Let A be a finite subset of an abelian group \mathbf{G} with the group operation \times . Then for any integer $k \geq 1$ one has*

$$\|A\|_{\mathcal{U}^k} \geq E^\times(A)^{2^k-k-1} |A|^{-(3 \cdot 2^k - 4k - 4)}.$$

Now let us prove a lower bound for $|kA|$, where A has small product/quotient set. The obtained estimate gives us a connection between the size of sumsets of a set and Gowers norms of its characteristic function.

Proposition 14 *Let $A \subset \mathbb{R}$ be a finite set, and k be a positive integer. Then*

$$|2^k A|^2 \gg_k \|A\|_{\mathcal{U}^{k+1}} \cdot \log^{-k} |A|. \quad (24)$$

Proof. We follow the arguments from [8]. Let us use the induction. The case $k = 1$ was obtained in [20], so assume that $k > 1$. Put $L = \log |A|$.

Without loss of generality one can suppose that $0 \notin A$. Taking any subset $S = \{s_1 < s_2 < \dots < s_r\}$ of A/A , we have by the main argument of [8]

$$|2^k A|^2 \geq \sum_{j=1}^{r-1} |2^{k-1} A_{s_j}| |2^{k-1} A_{s_{j+1}}|. \quad (25)$$

Now let S be a subset of A/A such that $\sum_{s \in S} |2^{k-1} A_s|^2 \gg_k L^{-1} \sum_s |2^{k-1} A_s|^2$ and for any two numbers s, s' the quantities $|2^{k-1} A_s|, |2^{k-1} A_{s'}|$ differ at most twice on S . Clearly, such S exists by the pigeonhole principle. Further, put $\Delta = \min_{s \in S} |2^{k-1} A_s|$. Thus, putting the set S into (25), we get

$$|2^k A|^2 \gg_k \Delta \sum_{s \in S} |2^{k-1} A_s| \gg_k L^{-1} \sum_s |2^{k-1} A_s|^2.$$

Now by the induction hypothesis and formula (23), we see that

$$|2^k A|^2 \gg_k L^{-k} \sum_s \|A_s\|_{\mathcal{U}^k} = L^{-k} \|A\|_{\mathcal{U}^{k+1}}.$$

This completes the proof. □

Proposition above has an immediate consequence.

Corollary 15 *Let $A \subset \mathbb{R}$ be a finite set, and k be a positive integer. Let also $M \geq 1$, and*

$$|AA| \leq M|A| \quad \text{or} \quad |A/A| \leq M|A|. \quad (26)$$

Then

$$|2^k A|^2 \gg_k |A|^{1+k/2} M^{-u_k} \cdot \log^{-k/2} |A|, \quad (27)$$

where

$$u_k = 2^k - k/2 - 1.$$

Proof. Combining Proposition 14 and Corollary 15, we obtain

$$|2^k A|^2 \gg_k \log^{-k} |A| \cdot E^\times(A)^{2^{k+1}-k-2} |A|^{-(3 \cdot 2^{k+1}-4k-8)}. \quad (28)$$

By assumption (26) and the Cauchy–Schwarz inequality (4), we get $E^\times(A) \geq |A|^3/M$. Substituting the last bound into (28), we have

$$|2^k A|^2 \gg_k \log^{-k} |A| \cdot |A|^{k+2} M^{-(2^{k+1}-k-2)}$$

as required. \square

Thus, for $|AA| \ll |A|$ or $|A/A| \ll |A|$, we have, in particular, that $|4A| \gtrsim |A|^2$. Actually, a stronger bound takes place. We thank to S.V. Konyagin for pointed this fact to us.

Corollary 16 *Let $A \subset \mathbb{R}$ be a finite set with $|A/A| \ll |A|$. Then*

$$|4A| \gtrsim |A|^{2+c},$$

where $c > 0$ is an absolute constant.

Proof. Without loss of generality one can suppose that $0 \notin A$. We use the arguments and the notation of the proof of Proposition 14. By formula (25), we have

$$|4A|^2 \geq \sum_{j=1}^{r-1} |A_{s_j} + A_{s_j}| |A_{s_{j+1}} + A_{s_{j+1}}|. \quad (29)$$

By Theorem 11 from [16] for any finite $B \subset \mathbb{R}$ one has $|B + B| \gtrsim_{M(B)} |B|^{3/2+c}$, where $c > 0$ is an absolute constant. Choose our set S such that $\sum_{s \in S} |A_s|^{3+2c} \gtrsim \sum_s |A_s|^{3+2c}$ and for any two numbers s, s' the quantities $|A_s|, |A_{s'}|$ differ at most twice on S . Clearly, such S exists by the pigeonhole principle. Further, put $\Delta = \min_{s \in S} |A_s|$. By the Hölder inequality and our assumption $|A/A| \ll |A|$ one has $\sum_s |A_s|^{3+2c} \gg |A|^{4+2c}$ and hence $\Delta \gg |A|$. It follows that $M(A_s) \ll 1$ for any $s \in S$ (see the definition of the quantity $M(A_s)$ in (7)). Applying Theorem 11 from [16] for sets A_{s_j} , combining with (29) and the previous calculations, we obtain

$$|4A|^2 \gtrsim \sum_{s \in S} |A_s|^{3+2c} \gtrsim \sum_s |A_s|^{3+2c} \gg |A|^{4+2c}.$$

This completes the proof. \square

The proof of our last proposition of this paper uses the same idea as the arguments of Theorem 12 and improves symmetric case of Lemma 33 from [18] for small M . The result is simple but it shows that for any set with small $|AA|$ or $|A/A|$ there is a "coset" splitting, similar to multiplicative subgroups in \mathbb{F}_p^* .

Proposition 17 *Let p be a prime number and $A \subseteq \mathbb{F}_p$ be a set, $|AA| \ll p^{2/3}$. Put $|AA| = M|A|$. Then*

$$\max_{x \neq 0} |A \cap (A + x)| \ll M^{9/4} |A|^{3/4}. \quad (30)$$

If $|A/A| = M|A|$ and $M^4|A|^3 \ll p^2$ then

$$\max_{x \neq 0} |A \cap (A + x)| \ll M^3 |A|^{3/4}. \quad (31)$$

Proof. Without loss of generality one can suppose that $0 \notin A$. Let $\Pi = AA$, $Q = A/A$. First of all, let us prove (30). It is easy to see that for any $x \in \mathbb{F}_p^*$ the following holds

$$(A \circ A)(x) \leq (\Pi \circ \Pi)(x/a) \quad \text{for all } a \in A. \quad (32)$$

Hence

$$(A \circ A)^2(x) \leq |A|^{-1} \sum_{a \in A} (\Pi \circ \Pi)^2(x/a) \leq |A|^{-1} \sum_a (\Pi \circ \Pi)^2(a) = |A|^{-1} \mathbf{E}^+(\Pi). \quad (33)$$

By Lemma 3 there is $A' \subseteq A$, $|A'| \geq |A|/2$ such that $|A'\Pi| \ll M^2|A|$. In particular, $|\Pi||A'||A'\Pi| \ll M^3|A|^3 \ll p^2$. Using Theorem 9 with $A = C = \Pi$ and $B = A'$, we get

$$\mathbf{E}^+(\Pi) \ll M^{9/2} |A|^{5/2}.$$

Combining the last bound with (33), we obtain (30).

To prove (31), note that the following analog of formula (32) takes place

$$(A \circ A)(x) \leq (Q \circ Q)(ax) \quad \text{for all } a \in A \quad (34)$$

and we can apply the previous arguments. In the situation by formula (5) of Lemma 3 one has $|QA| \leq M^3|A|$ and thus Theorem 9 with $A = C = Q$ and $B = A$ gives us

$$|A| \cdot \max_{x \neq 0} (A \circ A)^2(x) \leq \mathbf{E}^+(Q) \ll M^6 |A|^{5/2}.$$

This completes the proof. \square

References

- [1] A. BUSH, E. CROOT, *Few products, many h -fold sums*, arXiv:1409.7349v4 [math.CO] 18 Oct 2014.
- [2] M-C. CHANG, *Erdős-Szemerédi problem on sum set and product set*, Annals of Math. **157** (2003), 939–957.
- [3] G. ELEKES, *On the number of sums and products*, Acta Arith. **81** (1997), 365–367.
- [4] G. ELEKES, I. RUZSA, *Few sums, many products*, Studia Sci. Math. Hungar. **40**:3, (2003), 301–308.
- [5] P. ERDÖS, E. SZEMERÉDI, *On sums and products of integers*, Studies in pure mathematics, 213–218, Birkhäuser, Basel, 1983.
- [6] W.T. GOWERS, *A new proof of Szemerédi’s theorem*, GAFA, **11** (2001), 465–588.
- [7] L. LI, O. ROCHE-NEWTON, *Convexity and a sum-product type estimate*, Acta Arithmetica **156**(3) (2012), 247–256.
- [8] S.V. KONYAGIN, *h -fold Sums from a Set with Few Products*, MJCNT, **4**:3 (2014), 14–20.
- [9] S.V. KONYAGIN, I.D. SHKREDOV, *On sum sets of sets, having small product sets*, Transactions of Steklov Mathematical Institute, **3**:290 (2015), 304–316.
- [10] S.V. KONYAGIN, I.D. SHKREDOV, *New results on sum-products in \mathbb{R}* , Transactions of Steklov Mathematical Institute, accepted, arXiv:1602.03473v1 [math.CO].
- [11] O. ROCHE-NEWTON, *A short proof of a near-optimal cardinality estimate for the product of a sum set*, arXiv:1502.05560v1 [math.CO] 19 Feb 2015.
- [12] O. ROCHE-NEWTON, M. RUDNEV, I. D. SHKREDOV, *New sum-product type estimates over finite fields*, Advances in Mathematics, **293** (2016), 589–605.
- [13] M. RUDNEV, *On the number of incidences between planes and points in three dimensions*, preprint arXiv:1407.0426v3 [math.CO] 23 Dec 2014.
- [14] T. SCHOEN, I.D. SHKREDOV, *Higher moments of convolutions*, J. Number Theory **133**:5 (2013), 1693–1737.
- [15] I.D. SHKREDOV, *Energies and structure of additive sets*, Electronic Journal of Combinatorics, **21**:3 (2014), #P3.44, 1–53.
- [16] I.D. SHKREDOV, *On sums of Szemerédi-Trotter sets*, Transactions of Steklov Mathematical Institute, **289** (2015), 300–309.
- [17] I.D. SHKREDOV, *On tripling constant of multiplicative subgroups*, arXiv:1504.04522v1.
- [18] I.D. SHKREDOV, *Difference sets are not multiplicatively closed*, arXiv:1602.02360v2 [math.NT] 14 Feb 2016.

- [19] J. SOLYMOSI, *On the number of sums and products*, Bull. London Math. Soc., **37**:4 (2005), 491–494.
- [20] J. SOLYMOSI, *Bounding multiplicative energy by the sumset*, Advances in Mathematics Volume 222, Issue 2 (2009), 402–408.
- [21] E. SZEMERÉDI, W. T. TROTTER, *Extremal problems in discrete geometry*, Combinatorica **3** (1983), 381–392.
- [22] T. TAO AND V. VU, *Additive Combinatorics*, Cambridge University Press (2006).

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